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Control of Waves: Theory and Numerics

Enrique Zuazua
Universidad Autónoma
28049 Madrid, Spain
enrique.zuazua@uam.es
http://www.uam.es/enrique.zuazua

Guión:

- MOTIVACIÓN
- ELEMENTOS DE LA TEORÍA MATEMÁTICA
- FANTASMAS COMPUTACIONALES
- RETOS Y PERSPECTIVA.

MOTIVACIÓN

La misión de las científicas y científicos es comprender la realidad y en la medida de lo posible ayudar a transformarla en beneficio de su especie. Este principio que parece simple se transforma en un conjunto de procesos de gran complejidad en las sociedades desarrolladas. Comprender la realidad exige investigarla y ello supone diseñar un plan para hacerlo, obtener los recursos, desarrollar el plan y si todo va bien producir resultados reproducibles por otros grupos de investigación.

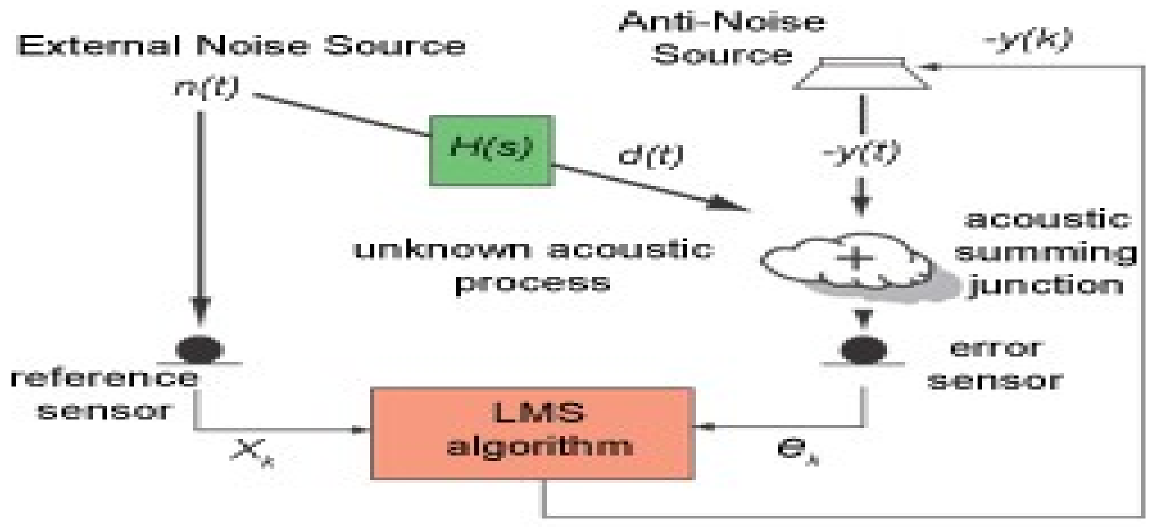
El País Digital

21 Septiembre, 2004

IS THE CONTROL OF WAVES AND, MORE PARTICULARLY, OF THE WAVE EQUATION RELEVANT?

The answer is, definitely, YES.

- Noise reduction in cavities and vehicles.



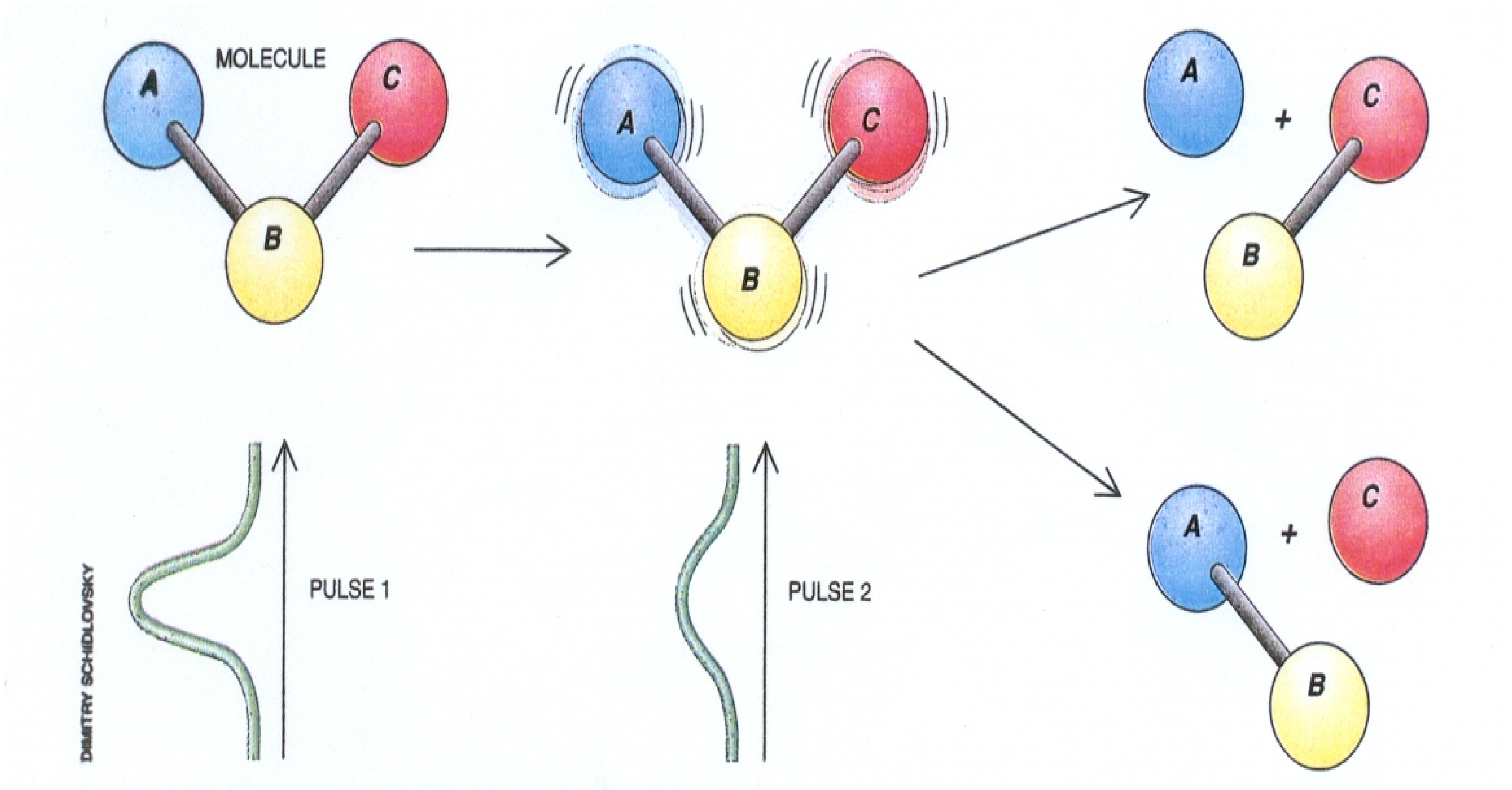
Closed-loop control diagram.

http://www.ind.rwth-aachen.de/research/noise_reduction.html

- Quantum control and Computing.

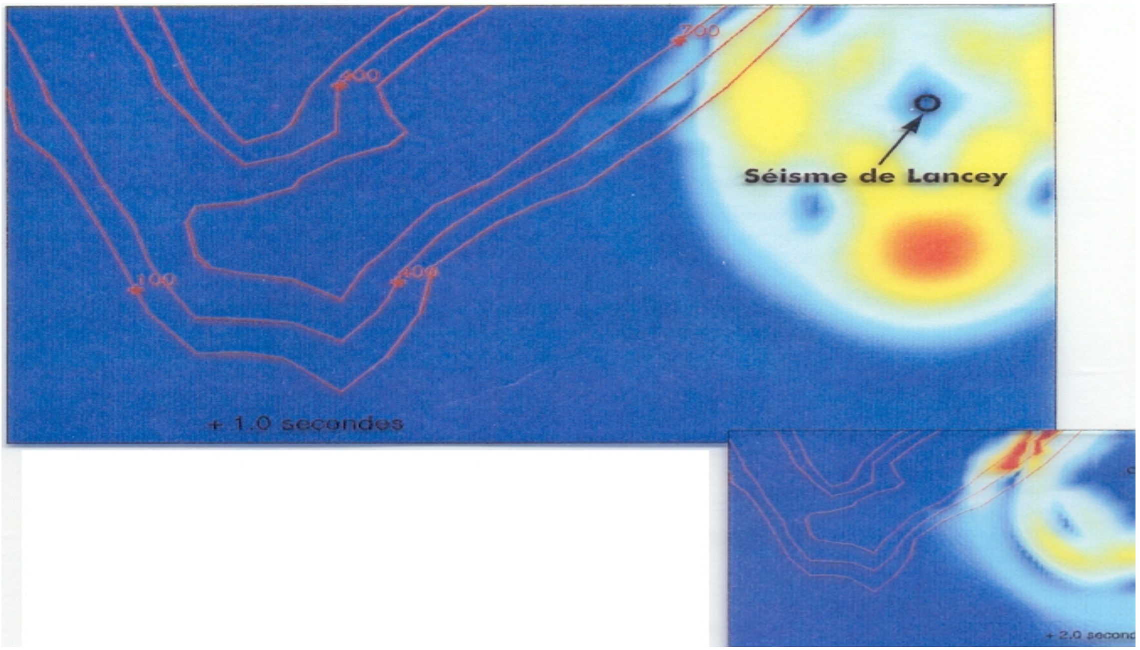
  Laser control in Quantum mechanical and molecular systems to design coherent vibrational states.

  In this case the fundamental equation is the Schrödinger one. Most of the theory we shall develop here applies in this case too. The Schrödinger equation may be viewed as a wave equation with inifnite speed of propagation.

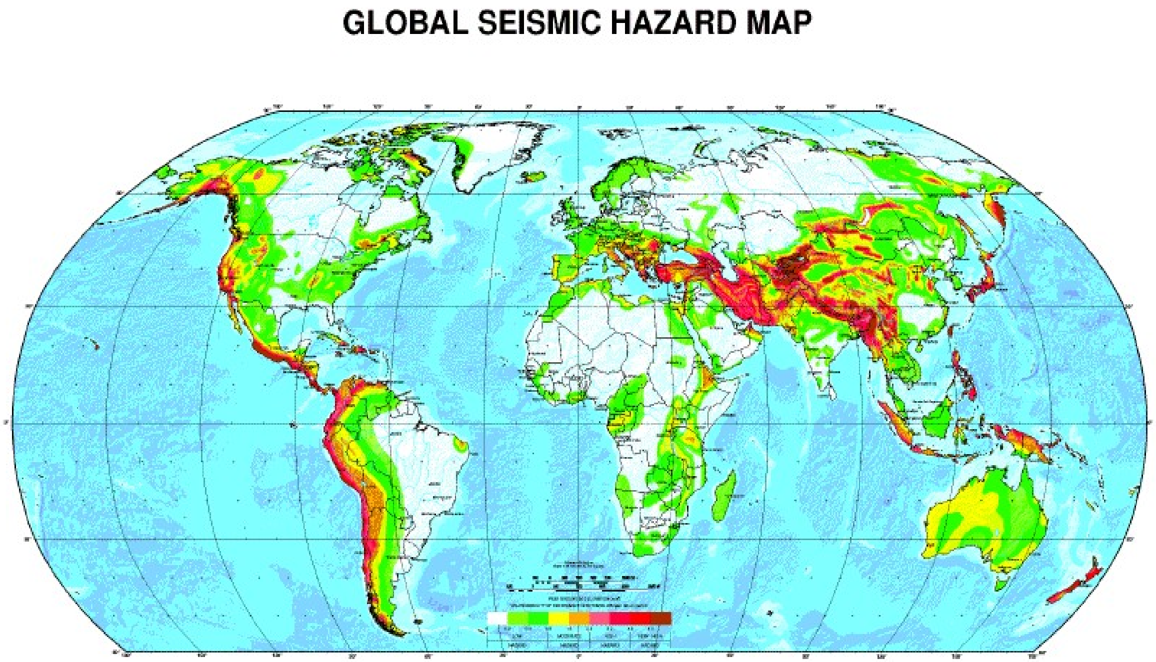


P. Brumer and M. Shapiro, Laser Control of Chemical reactions, Scientific American, March, 1995, pp.34-39.

- Seismic waves, earthquakes.

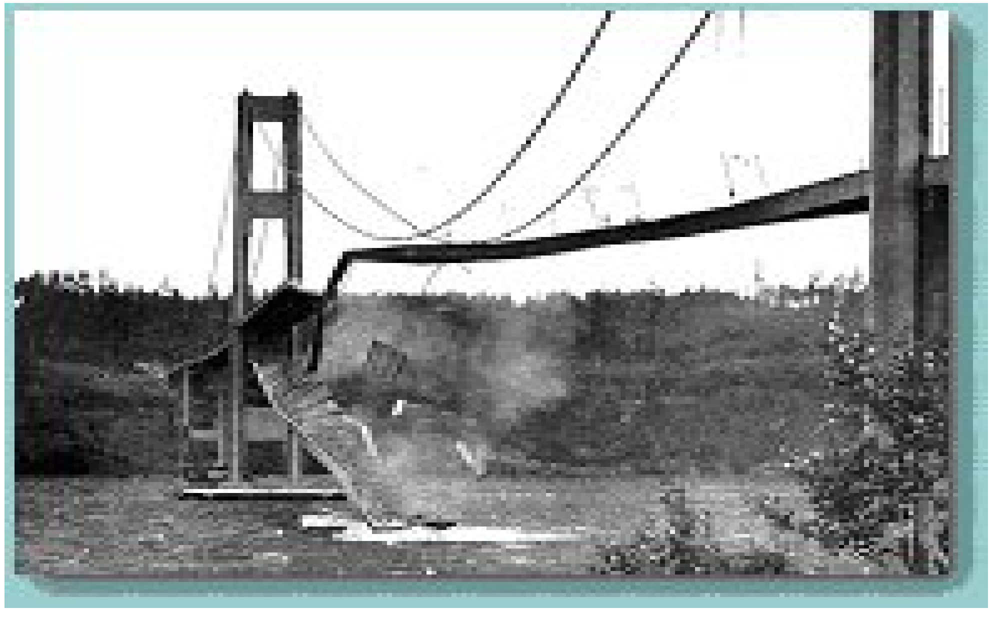


F. Cotton, P.-Y. Bard, C. Berge et D. Hatzfeld, Qu'est-ce qui fait vibrer Grenoble?, La Recherche, 320, Mai, 1999, 39-43.



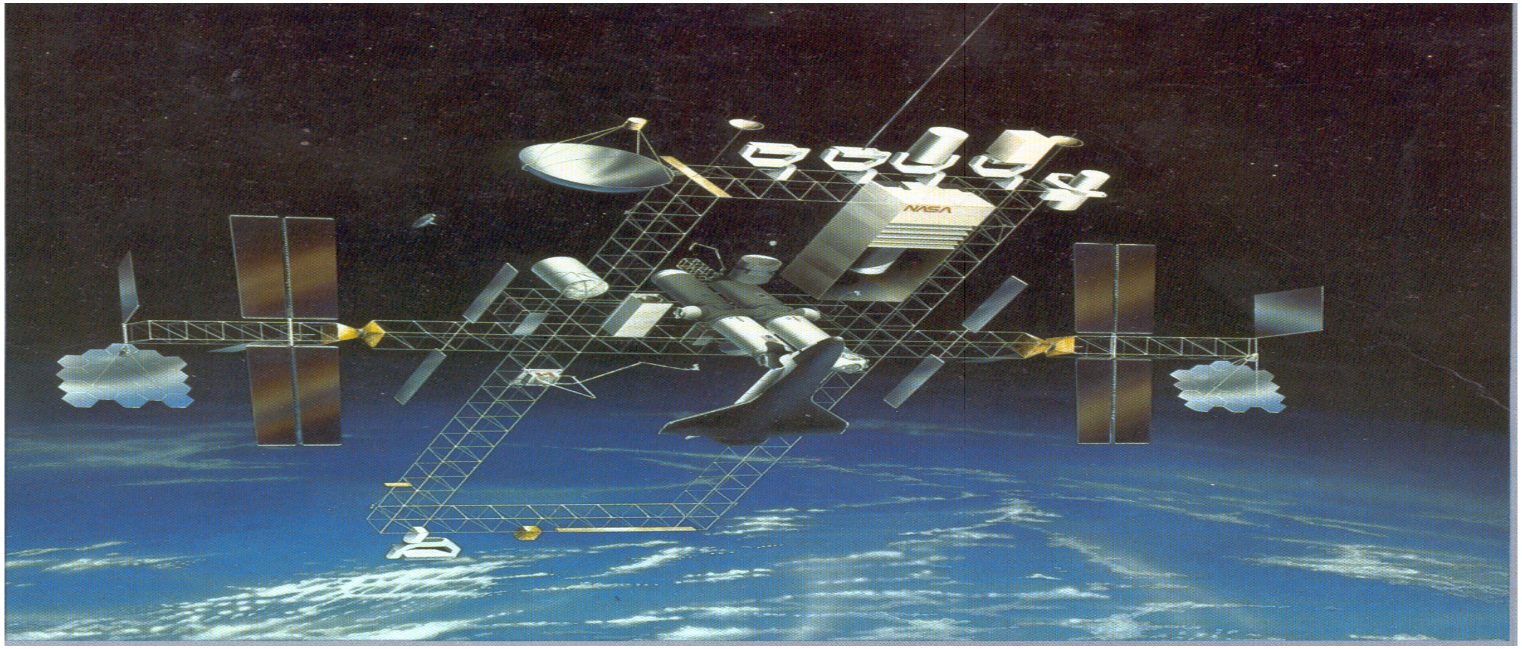
http://earthquake.usgs.gov/hazards/probability.html

- Flexible structures.



Takoma, USA, 1940

http://astro.if.ufrgs.br/evol/takoma.htm

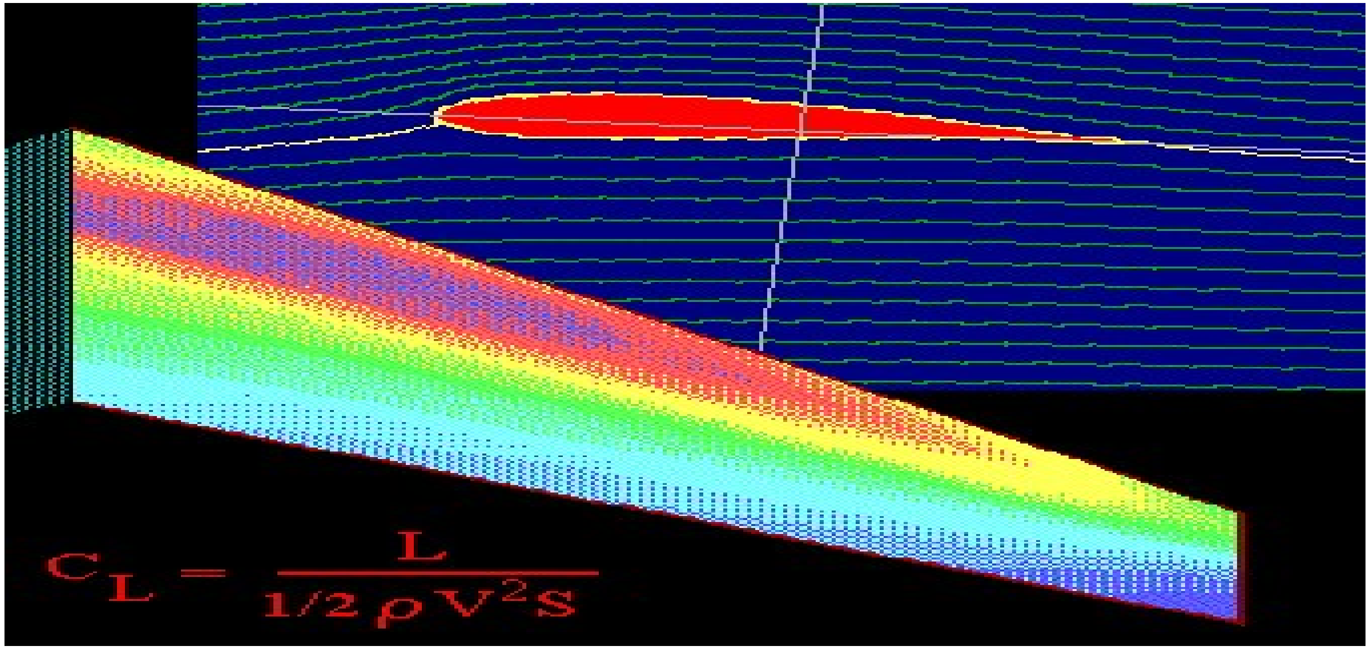


SIAM Report on "Future Directions in Control Theory. A Mathematical Perspective", W. H. Fleming, ed., 1988.

- Environment.

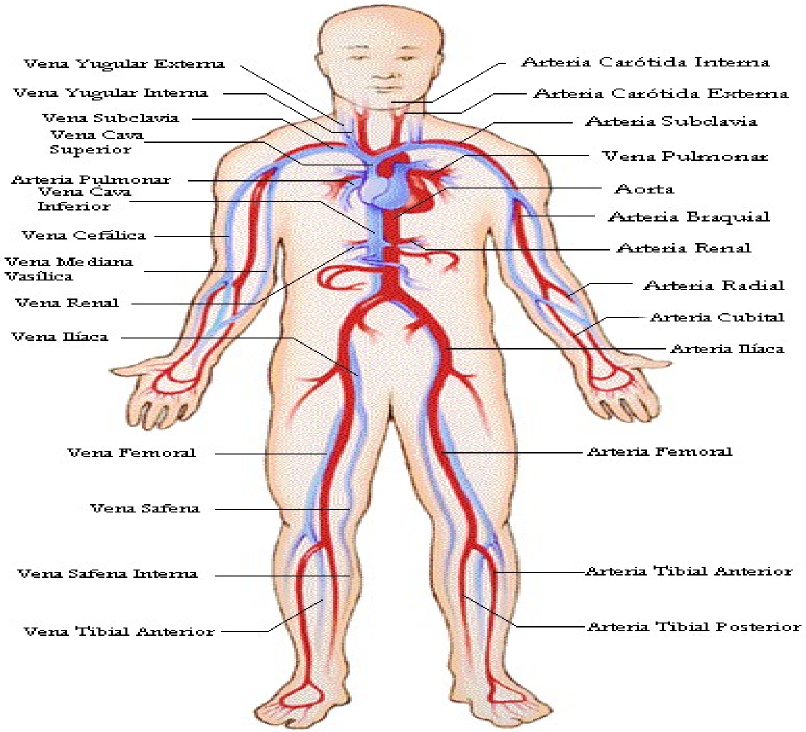
The Thames barrier.

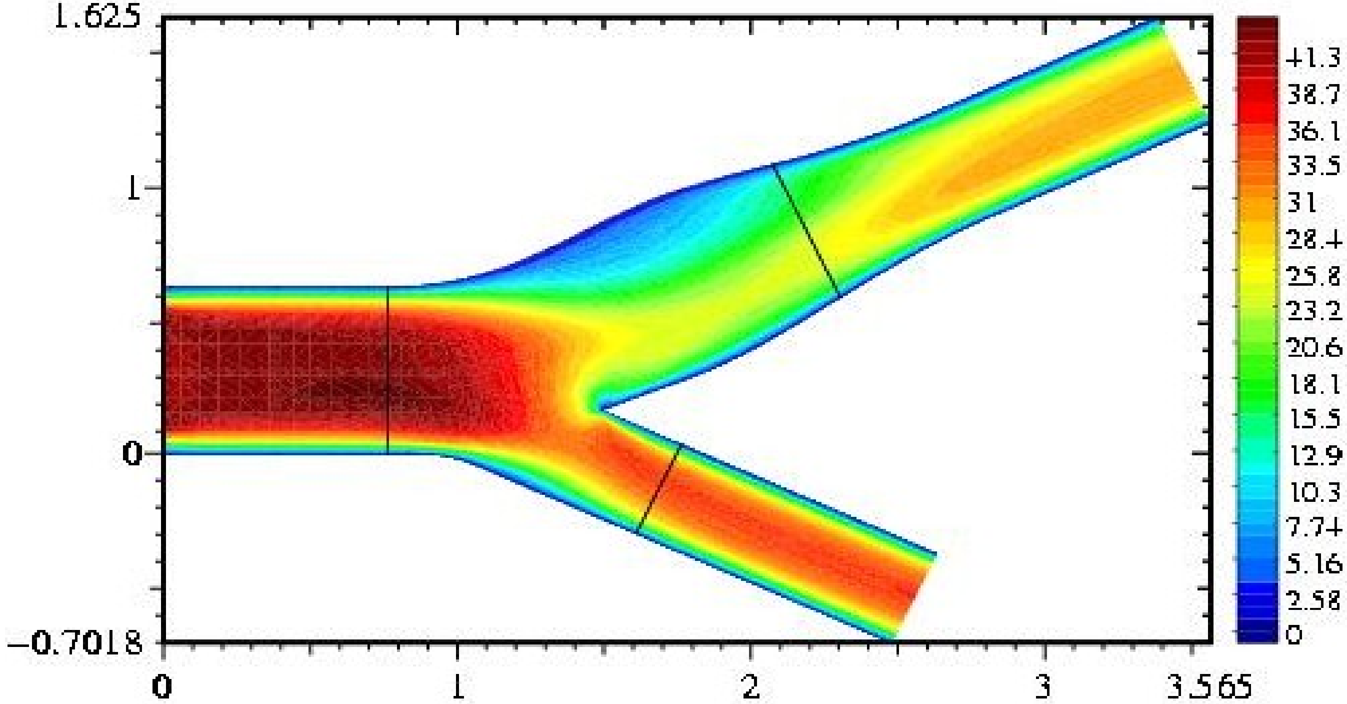
• Optimal shape design in aeronautics.



Optimal shape design of a "wing" within an Euler flow, for drag reduction.

- El sistema cardiovascular humano





1.625
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31
28.4
25.8
23.2
20.6
18.1
15.5
12.9
10.3
7.74
5.16
2.58
0

El bypass.

ELEMENTOS DE LA TEORÍA MATEMÁTICA

THE 1-D CONTROL PROBLEM

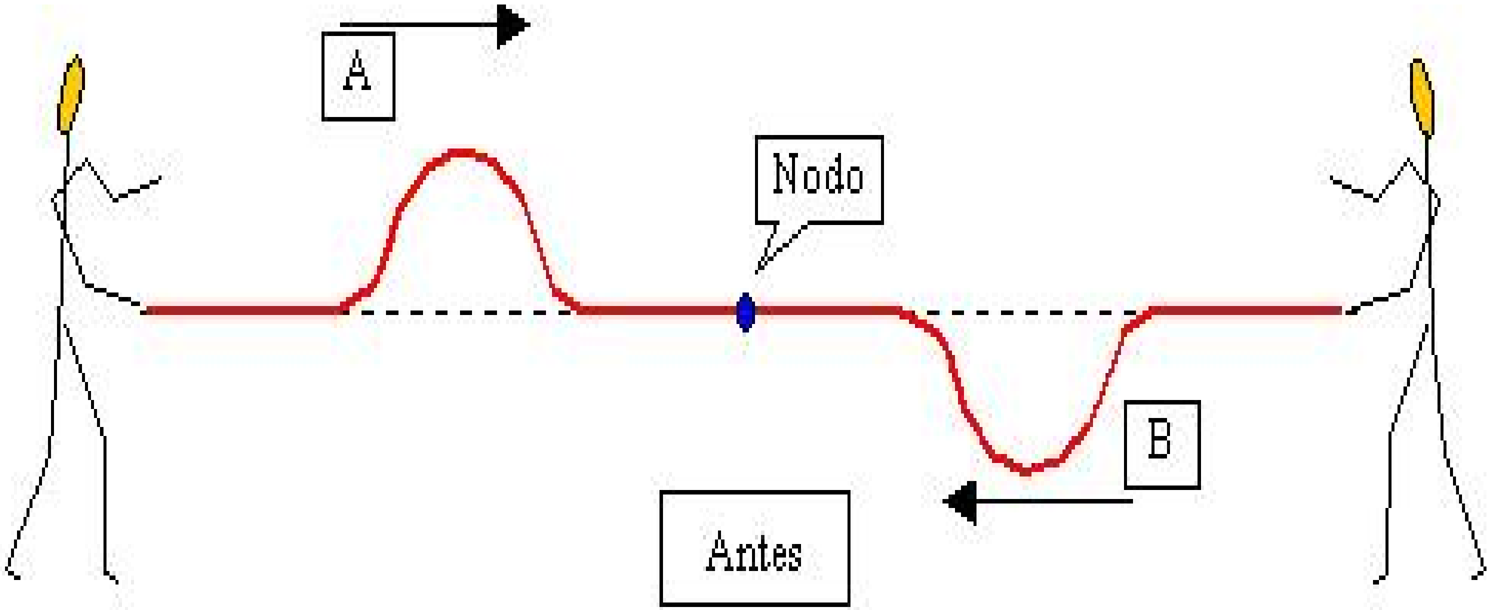
The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control one one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.

The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1.$$


A
Nodo
B
Antes

THE 1-D OBSERVATION PROBLEM

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & 0 < x < 1. \end{cases}$$

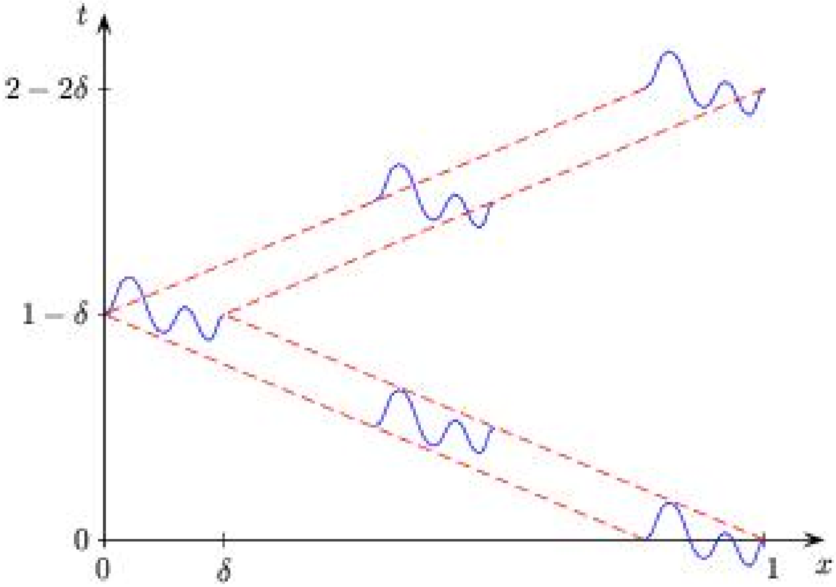
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_x(x,t)|^2 + |u_t(x,t)|^2 \right] dx = E(0),\ \ \forall 0 \le t \le T.$$

The question is then reduced to analyze whether the folllowing inequality is true. This is the so called observability inequality:

$$E(0) \le C(T) \int_0^T |u_x(1,t)|^2 \, dt.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \geq 2$.



$$E(0) \leq C(T) \int_0^T |u_x(1,t)|^2 \, dt.$$

Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

This observability inequality is easy to prove by several means.

- Use D'Alambert's formula

$$u = f(x+t) + g(x-t)$$

  indicating that information propagates along rays with velocity one, and bounces on the boundary points.

- Use the Fourier representation of solutions in which it is clearly seen that solutions are periodic with time-period 2.

- Multipliers: Multiply the equation by xu_x, u_t and u and integrate by parts....

CONSTRUCTION OF THE CONTROL:

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = u_x(1,t),$$

where u is the solution of the adjoint system corresponding to initial data $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(u^0, u^1) = \frac{1}{2}\int_0^T |u_x(1,t)|^2 dt + \int_0^1 y^0 u^1 dx - < y^1, u^0 >_{H^{-1}\times H_0^1},$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that $u_x(1,t) \in L^2(0,T)$ (hidden regularity).

Moreover,

COERCIVITY OF J = OBSERVABILITY INEQUALITY.

CONCLUSION:

The 1-d wave equation is controllable from one end, in time 2, twice the length of the interval.

Similar results are true in several space dimensions. The region in which the observation/control applies needs to be large enough to capture all rays of Geometric Optics.

THE PROBLEM:

EFFICIENTLY COMPUTE NUMERICALLY THE CONTROL!

WARNING ! TWO DIFFERENT ISSUES:

When a continuous model, written in PDE terms, is controllable, two important issues arise in the context of Numerical Simulation:

- Efficiently compute numerically the control.

- To control a discrete model, a numerical discretized version of the continuous model.

Both problems are relevant, but they may provide different results.

Both approaches are often mixed in the literature (leading to uncertain results....)

A FACT

THE PROCESSES OF CONTROL AND NUMERICS DO NOT COMMUTE

$$\text{CONTROL+NUMERICS} \neq \text{NUMERICS+CONTROL}$$

FROM FINITE TO INFINITE DIMENSIONS IN PURELY CONSERVATIVE SYSTEMS.....

FANTASMAS NUMÉRICOS

THE SEMI-DISCRETE PROBLEM: $1-D$.

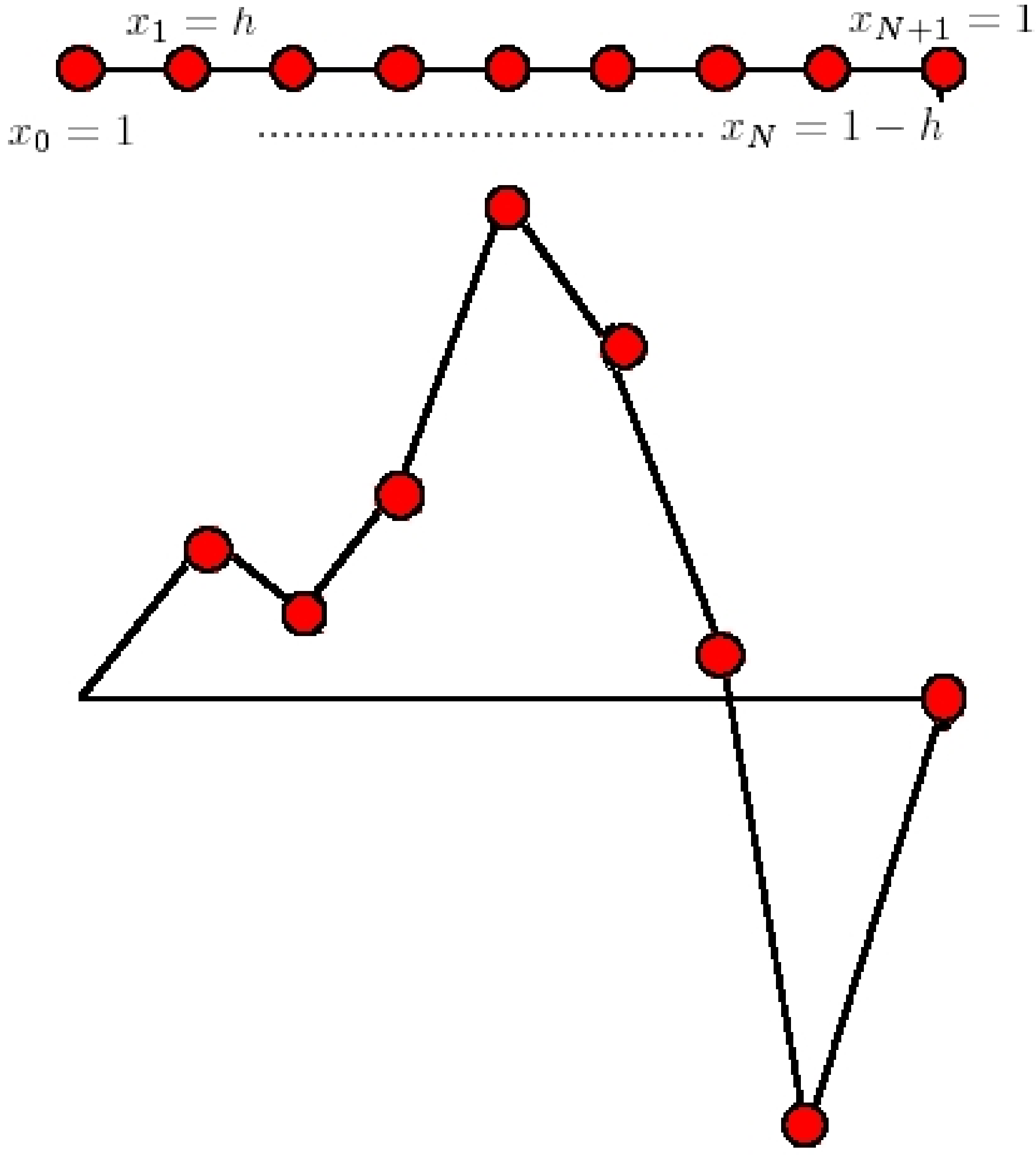
Set $h = 1/(N+1) > 0$ and consider the mesh

$$x_0 = 0 < x_1 < ... < x_j = jh < x_N = 1 - h < x_{N+1} = 1,$$

which divides $[0,1]$ into $N+1$ subintervals $I_j = [x_j, x_{j+1}]$, $j = 0, ..., N$.

Finite difference semi-discrete approximation of the wave equation:

$$\begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ u_j(t) = 0, & j = 0,\ N+1,\ 0 < t < T \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1, & j = 1, \dots, N. \end{cases}$$



$x_1 = h$
$x_{N+1} = 1$
$x_0 = 1$
$x_N = 1 - h$

The energy of the semi-discrete system (obviuosly a discrete version of the continuous one)

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[|\ u_j'\ |^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right].$$

It is constant in time.

Is the following observability inequality true?

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt$$

$$\left(-\frac{u_N(t)}{h} = \frac{u_{N+1} - u_N(t)}{h} \sim u_x(1,t). \right)$$

YES! It is true for all $h > 0$ and for all time T.

BUT, FOR ALL $T > 0$ (!!!!!)

$$C_h(T) \rightarrow \infty, \quad h \rightarrow 0.$$

THE FOLLOWING "INTUITIVE" CONJECTURE IS COMPLETELY FALSE:

* The constant $C_h(T)$ blows-up for $T < 2$ as $h \rightarrow 0$ since the inequality fails for the wave equation.

* The constant $C_h(T)$ remains bounded for $T \geq 2$ as $h \rightarrow 0$ and one recovers in the limit the observability inequality for the wave equation.

CONCLUSION

The classical convergence (consistency+stability) does not guarantee continuous dependence for the observation problem with respect to the discretization parameter.

WHY?

Convergent numerical schemes do reproduce all continuous waves but, when doing that, they create a lot of spurious (non-realistic, purely numerical) high frequency solutions. This spurious solutions distroy the observation properties and are an obstacle for the controls to converge as the mesh-size gets finer and finer.

SPECTRAL ANALYSIS

Eigenvalue problem

$$-\tfrac{1}{h^2}\left[w_{j+1} + w_{j-1} - 2w_j\right] = \lambda w_j,\ j = 1, \ldots, N$$
$$w_0 = w_{N+1} = 0.$$

The eigenvalues $0 < \lambda_1(h) < \lambda_2(h) < \cdots < \lambda_N(h)$ are

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$$

and the eigenvectors

$$w_k^h = \left(w_{k,1}, \ldots, w_{k,N}\right)^T : w_{k,j} = \sin(k\pi jh),\ k, j = 1, \ldots, N.$$

It follows that

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0$$

and the eigenvectors coincide with those of the wave equation.

Then, the solutions of the semi-discrete system may be written in Fourier series as follows:

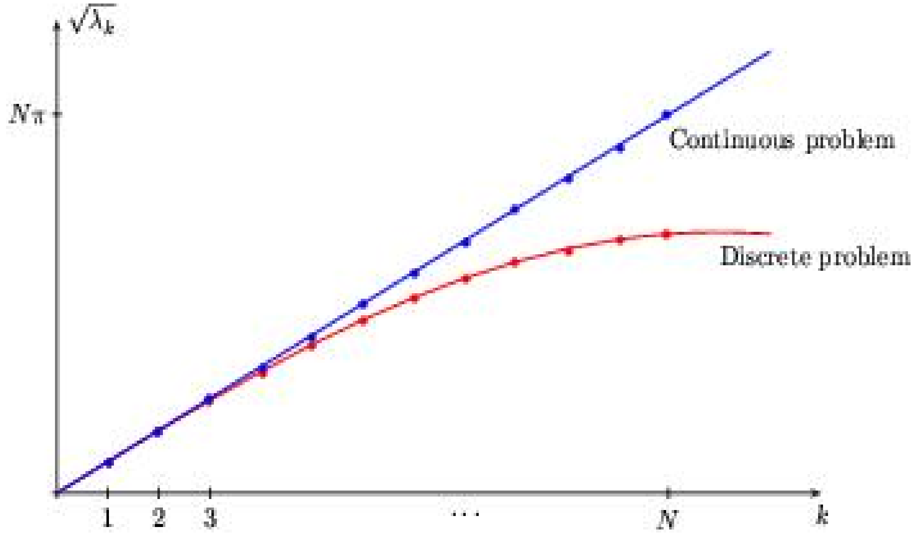
$$\vec{u} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right) \right) \vec{w}_k^h.$$

Compare with the Fourier representation of solutions of the continuous wave equation:

$$u = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x)$$

The only relevant difference is that the time-frequencies do not quite coincide, but they converge as $h \to 0$.

DISPERSION DIAGRAM: LACK OF GAP.



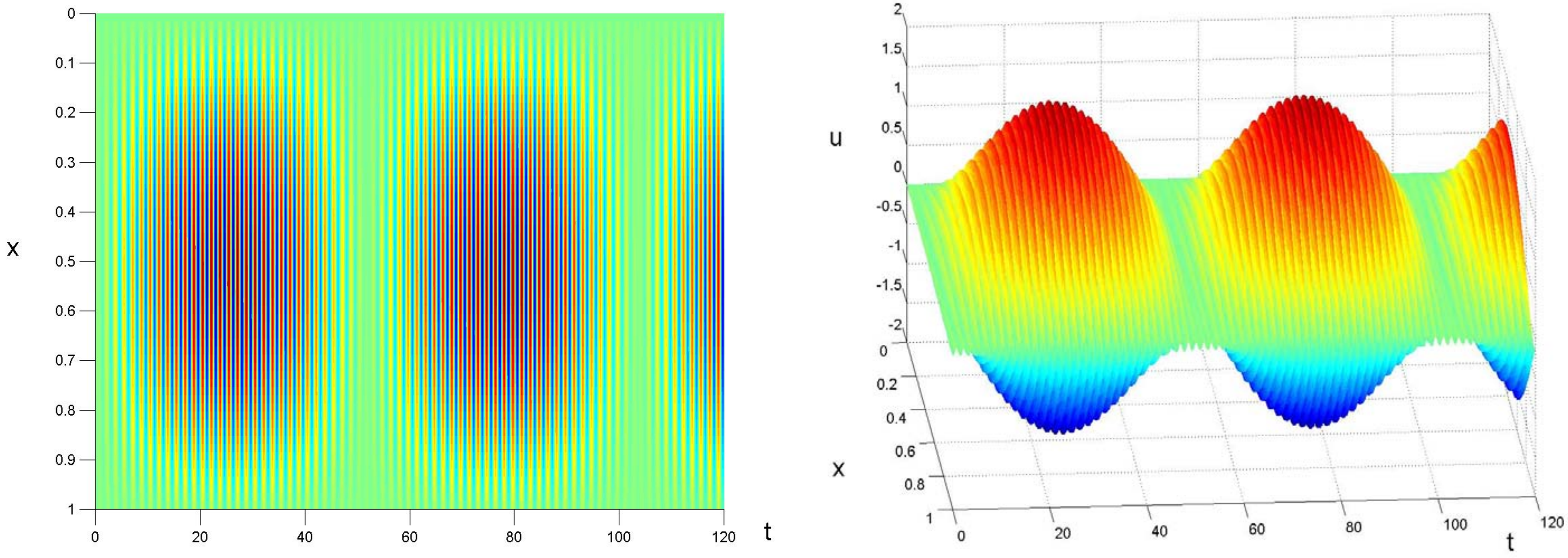
Graph of the square roots of the eigenvalues both in the continuous and in the discrete case. The gap is clearly independent of k in the continuous case while it is of the order of h for large k in the discrete one.

SPURIOUS NUMERICAL SOLUTION

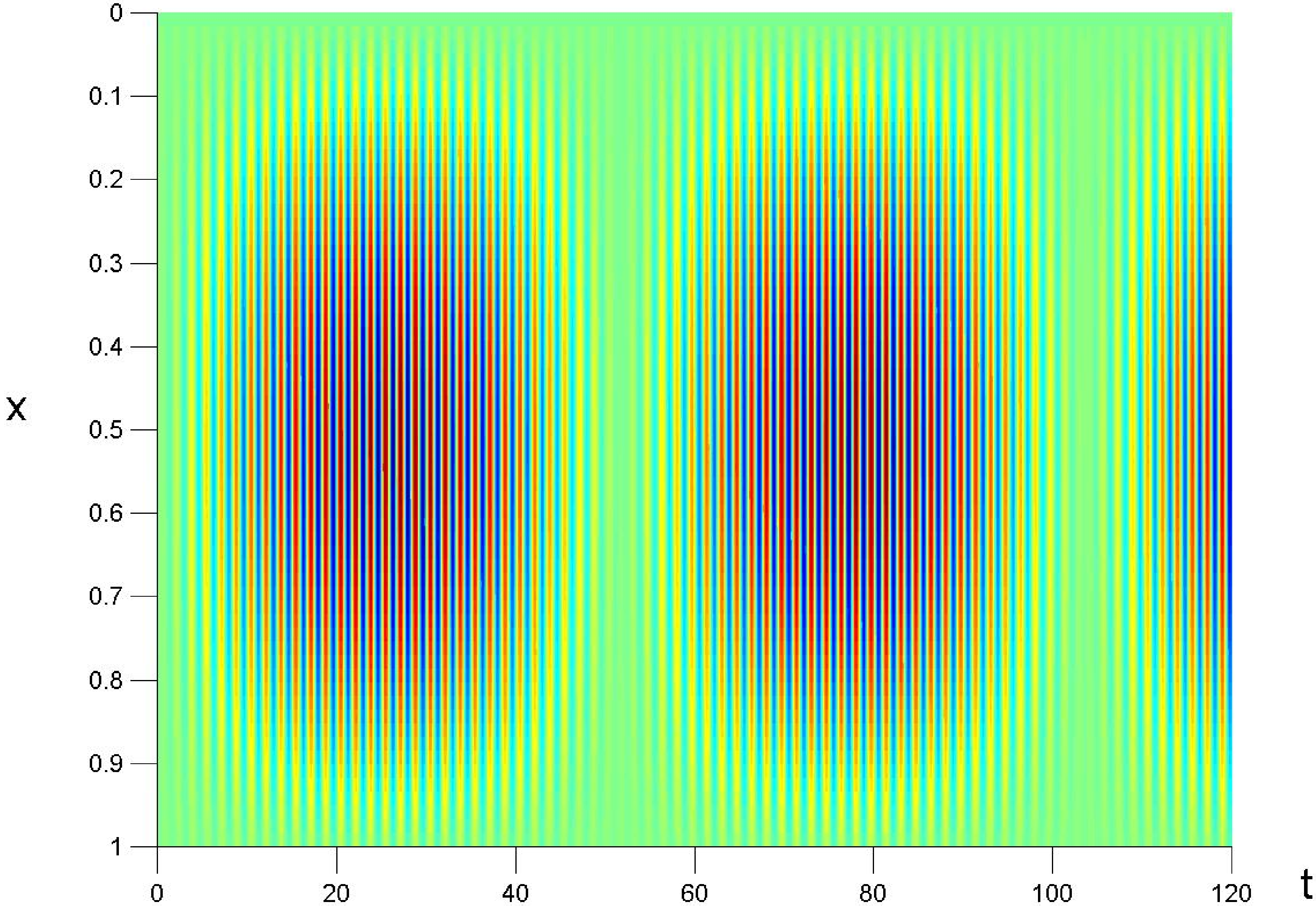
$$\vec{u} = \exp\left(i\sqrt{\lambda_N(h)}\,t\right)\vec{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}\,t\right)\vec{w}_{N-1}.$$

Spurious semi-discrete wave combining the last two eigenfrequencies with very little gap:

$$\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h.$$



$h = 1/61$, $(N = 60)$, $0 \leq t \leq 120$. *The solution exhibits a time-periodicity property with period τ of the order of $\tau \sim 50$ which contradicts the time-periodicity of period 2 of the wave equation. High frequency wave packets travel at a group velocity $\sim h$.*



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0.5
0.6
0.7
0.8
0.9
1
x
0
20
40
60
80
100
120
t

GAP

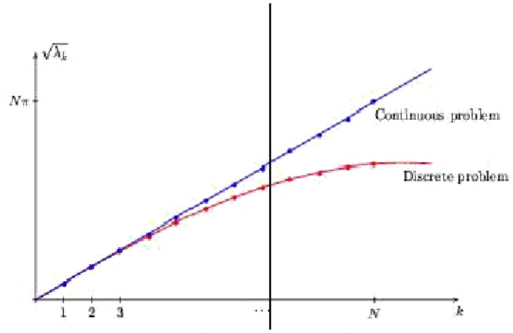
=

GROUP VELOCITY

=

VELOCITY OF PROPAGATION OF HIGH FREQUENCY WAVE PACKETS.

WHAT IS THE REMEDY?



filtering parameter:
$k \leq \delta N$, $0 < \delta < 1$

To filter the high frequencies, i.e. keep only the components of the solution corresponding to indexes: $k \leq \delta/h$ with $0 < \delta < 1$.

Filtering restablishes the gap condition, then waves propagate with a speed which is uniform with respect to h and the observability inequality becomes uniform too.

$$\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} \geq \pi \cos\left(\frac{\pi\delta}{2}\right) > 0, \text{ for } k \leq \delta h^{-1}.$$

This can be done rigorously with the aid of

Ingham's Theorem. (1936) *Let $\{\mu_k\}_{k\in\mathbf{Z}}$ be a sequence of real numbers such that*

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \forall k \in \mathbf{Z}.$$

Then, for any $T > 2\pi/\gamma$ there exists $C(T,\gamma) > 0$ such that

$$\frac{1}{C(T,\gamma)} \sum_{k\in\mathbf{Z}} | a_k |^2 \leq \int_0^T \left| \sum_{k\in\mathbf{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in\mathbf{Z}} | a_k |^2$$

for all sequences of complex numbers $\{a_k\} \in \ell^2$.

CONCLUSION.

Given any $T > 2$, choose $0 < \delta < 1$ such that

$$T > 2/\cos\left(\frac{\pi\delta}{2}\right) \quad \text{or} \quad \delta > \frac{2}{\pi}\arccos(2/T).$$

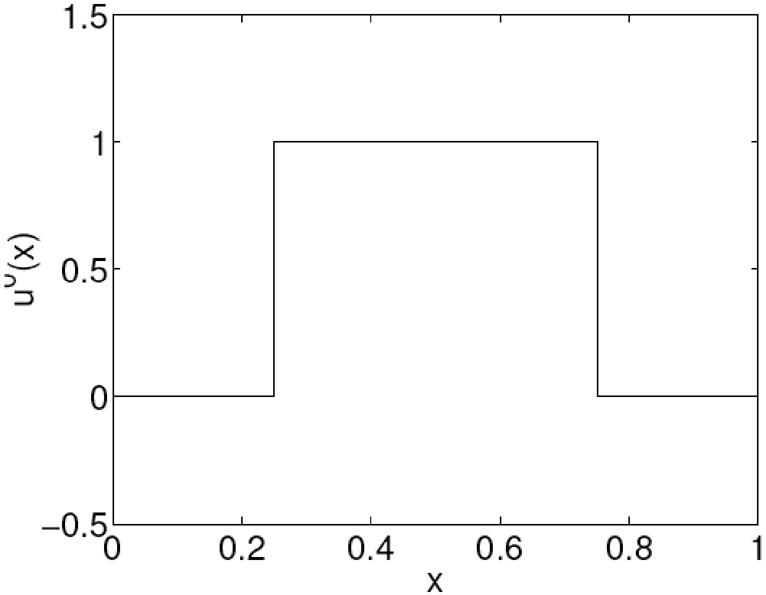
The choice of $0 < \delta < 1$ is obviously possible since $2/T < 1$.

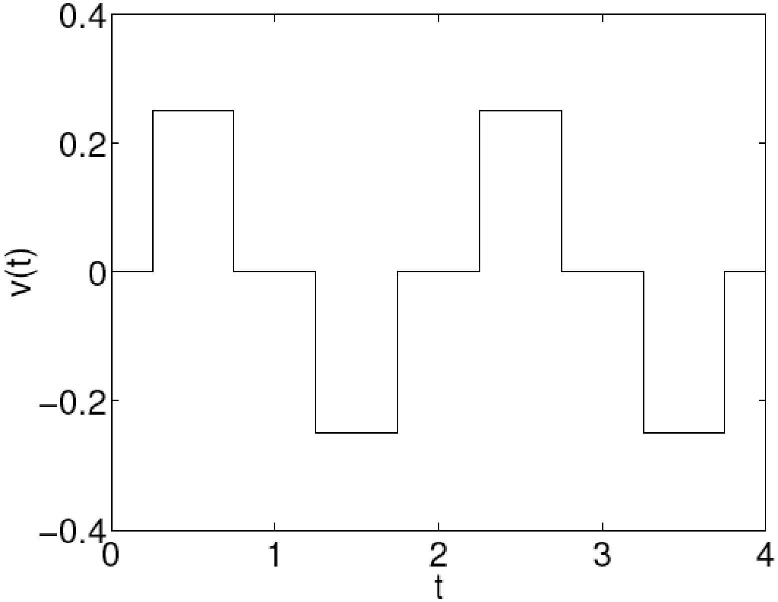
Then, we can control UNIFORMLY ON h the solution PARTIALLY:

$$\pi_\delta(y(T), y_t(T)) = 0$$

and

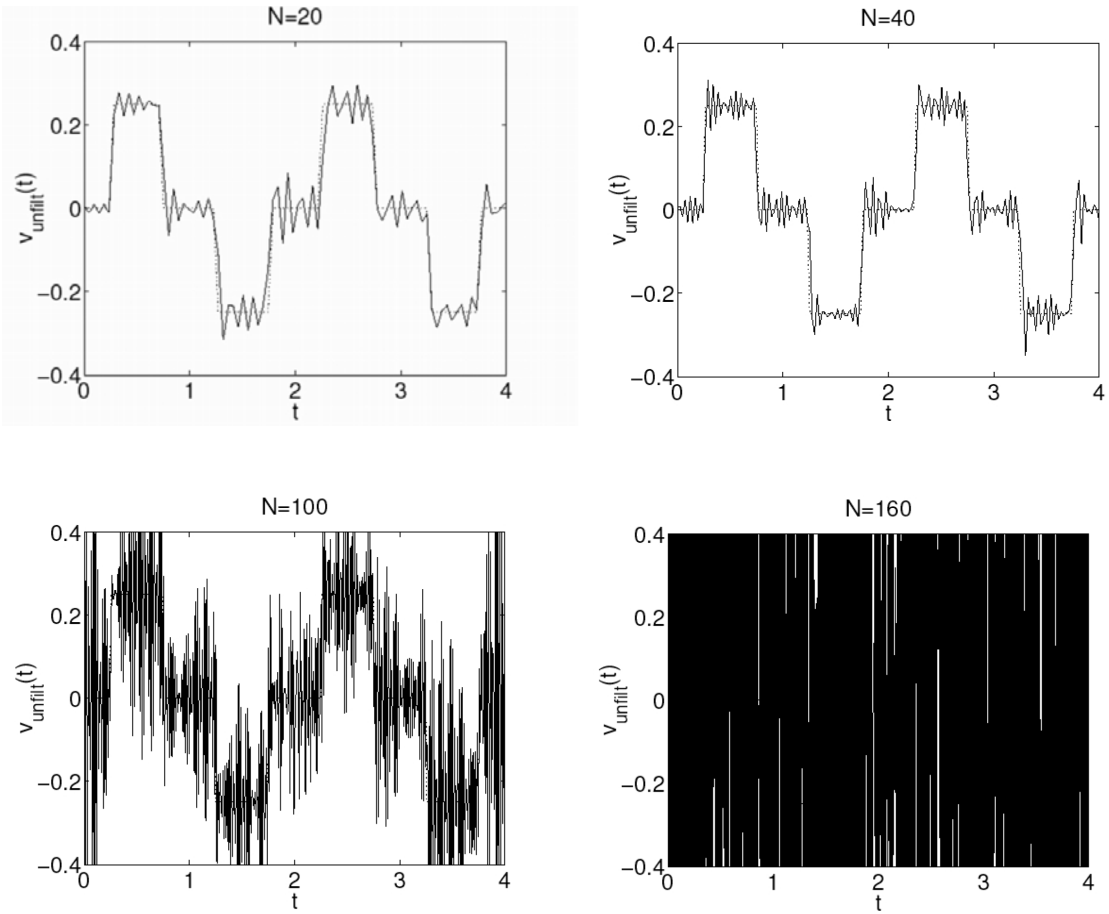
the numerical controls $\quad v_h \to v, \quad$ the control of the wave equation.



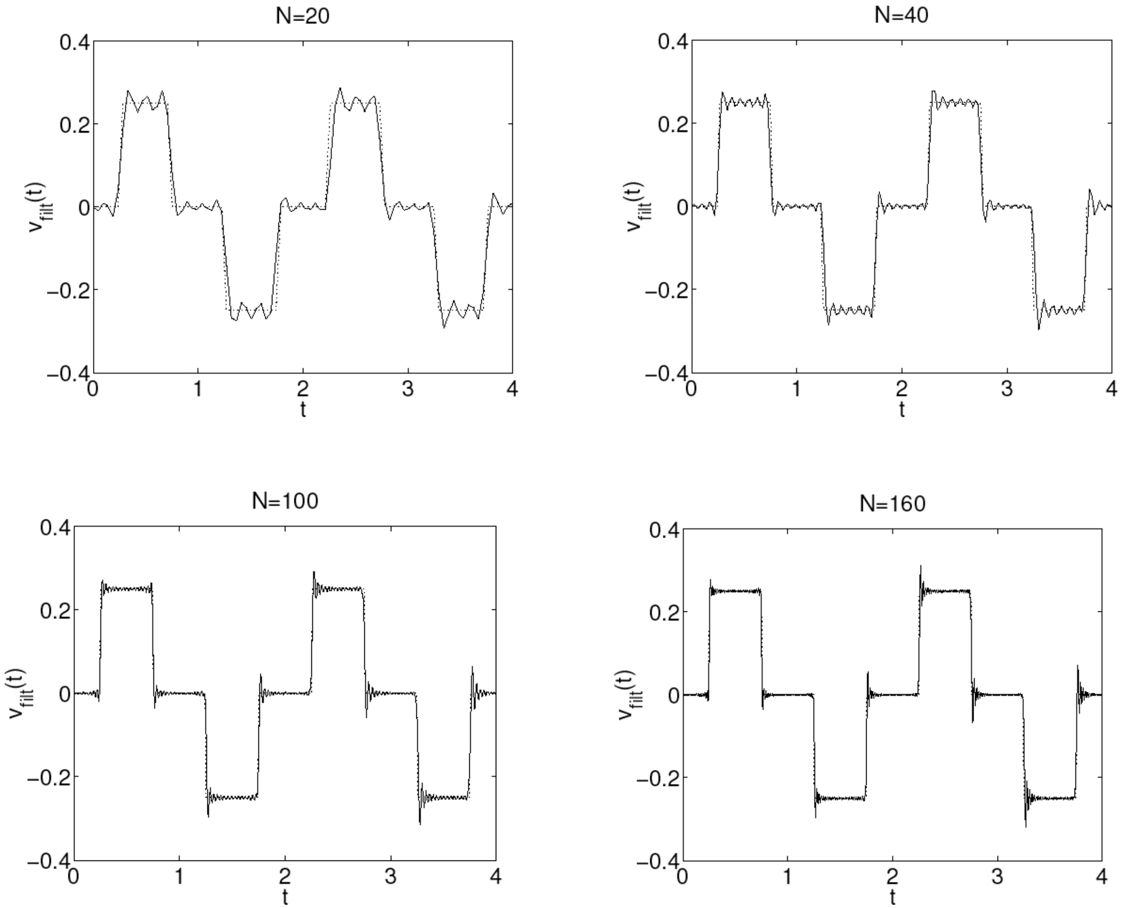


Plot of the initial datum to be controlled for the string occupying the space interval $0 < x < 1$.

Plot of the time evolution of the exact control for the wave equation in time $T = 4$.



Without filtering, the control diverges as $h \to 0$.



With appropriate filtering the control converges as $h \to 0$.

ULTIMATE GOAL

To develop a class of numerical schemes (new or not) for which the convergence of controls might be guaranteed a priori with minimal computational cost.

The most natural approaches (finite differences and FINITE ELEMENTS) do not work and they have to be complemented with other strategies:

* filtering of high frequencies,

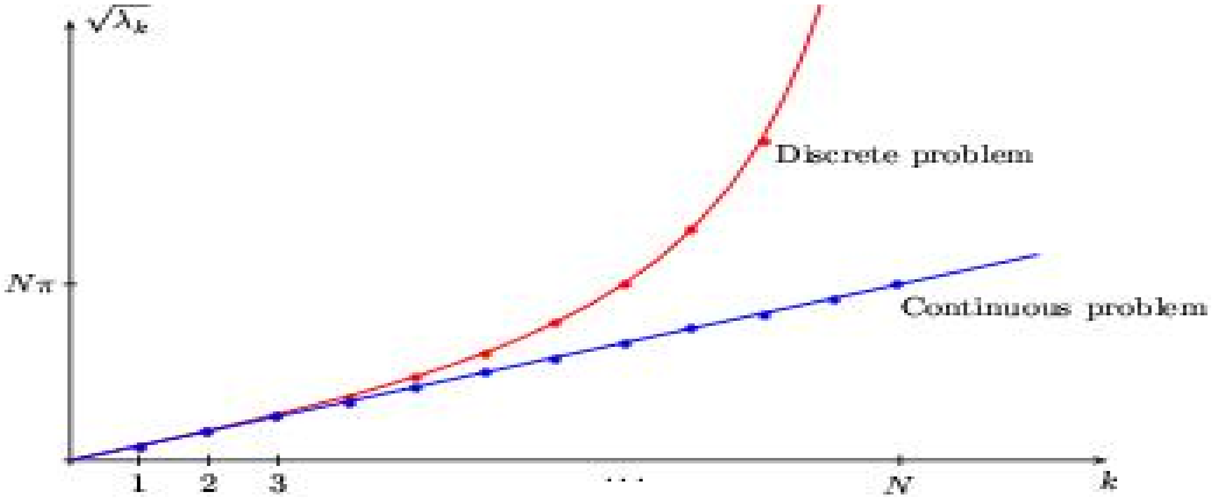
* mixed finite elements,

* multi-grid algorithms,

* wavelets,

* numerical viscosity,...

MIXED FINITE ELEMENTS



Square roots of the eigenvalues both in the continuous and in the discrete case with mixed finite elements. The gap of the discrete problem is uniform with respect to j *and* h *and, in fact, tends to infinity for the highest frequencies as* $h \to 0$.

THE CONTROL PROBLEM IN SEVERAL SPACE DIMENSIONS

The same problems arise in several space dimensions:

Let Ω be a bounded domain of $\mathbf{R}^n, n \geq 1$, with boundary Γ of class C^2. Let Γ_0 be an open and non-empty subset of Γ and $T > 0$.

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0,T) \\ y = v(x,t) 1_{\Gamma_0} & \text{on} \quad \Sigma = \Gamma \times (0,T) \\ (x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

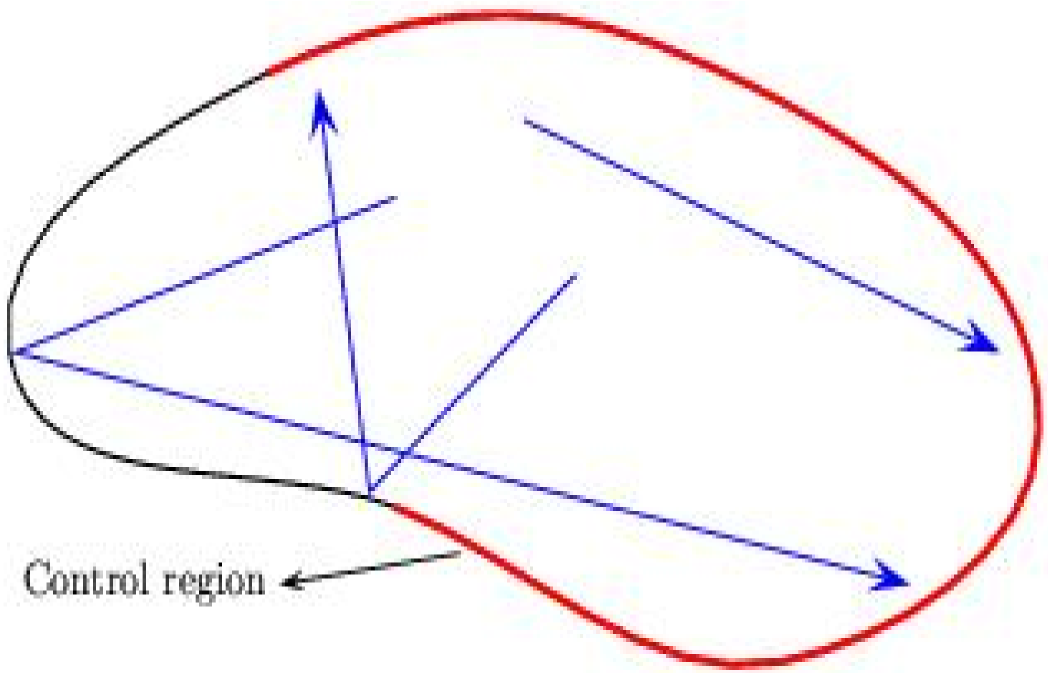
The problem of *controllability*, generally speaking, is as follows: *Given* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *find* $v \in L^2(\Gamma_0 \times (0,T))$ *such that the solution of system* (3.1) *satisfies*

$$y(T) \equiv y_t(T) \equiv 0.$$

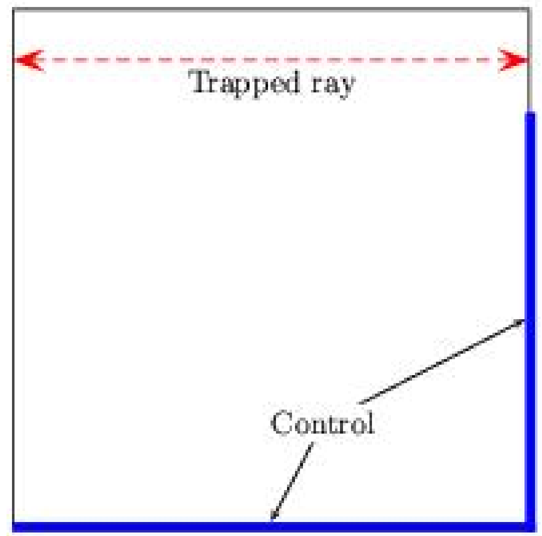
The answer is by now well known (Bardos-Lebeau-Rauch, Burq-Gérard, Ralston,....):

The wave equation is controllable from Γ_0 in time T if all rays of Geometric Optics intersect Γ_0 in a time less than T at a non-difractive point.

This statement is an extension of the one above on the 1-d wave equation. But this time the proof requires much more sophisticated tools: Microlocal analysis, the propagation of microlocal deffect measures,...



Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case.



The Geometric Control Condition is not satisfied, whatever $T > 0$ is, in the square domain when the control is located on a subset of two consecutive sides of the boundary, leaving a subsegment uncontrolled. There is an horizontal a ray that bounces back and forth for all time perpendicularly on two points of the vertical boundaries where the control does not act.

In all cases the control is equivalent to an observation problem for the adjoint wave equation:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad Q = \Omega \times (0,T) \\ u = 0 & \text{on} \quad \Sigma = \Gamma \times \times (0,T) \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases}$$

Is it true that:

$$E_0 \leq C(\Gamma_0, T) \int_{\Gamma_0} \int_0^T \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \quad ?$$

And a sharp discussion of this inequality requires of Microlocal analysis. Partial results may be obtained by means of multipliers: $x \cdot \nabla u$, u_t, $u, \ldots$

THE 5-POINT FINITE-DIFFERENCE SCHEME

$$u''_{j,k} - \frac{1}{h^2}\left[u_{j+1,k} + u_{j-1,k} - 4u_{j,k} + u_{j,k+1} + u_{j,k-1}\right] = 0.$$

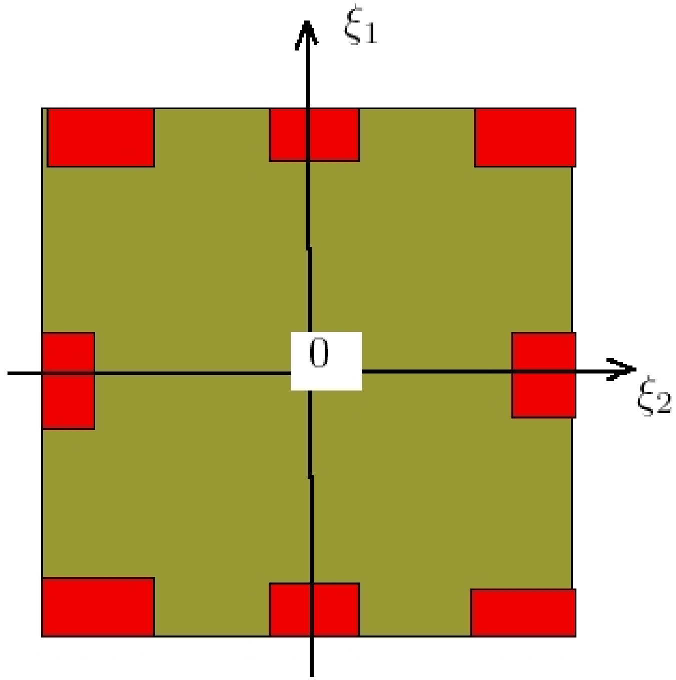
The energy of solutions is constant in time:

$$E_h(t) = \frac{h^2}{2}\sum_{j=0}^{N}\sum_{k=0}^{N}\left[\mid u'_{jk}(t)\mid^2 + \left|\frac{u_{j+1,k}(t) - u_{j,k}(t)}{h}\right|^2 + \left|\frac{u_{j,k+1}(t) - u_{j,k}(t)}{h}\right|^2\right].$$

Without filtering observability inequalities fail in this case too.

Understanding how filtering should be used requires of a microlocal Analysis of the propagation of numerical waves. F. Macià (Ph. D. Thesis, Madrid, 2002) combining von Neumann analysis and Wigner measures developments (P. Gérard, P. L. Lions & Th. Paul, G. Lebeau, ...). Most of the results that this analysis yields were previously predicted by N. Trefethen.

Through the von Neumann analysis one can define the discrete rays along which energy propagates. This indicates that the filtering has to be eprformed according to the following figure:



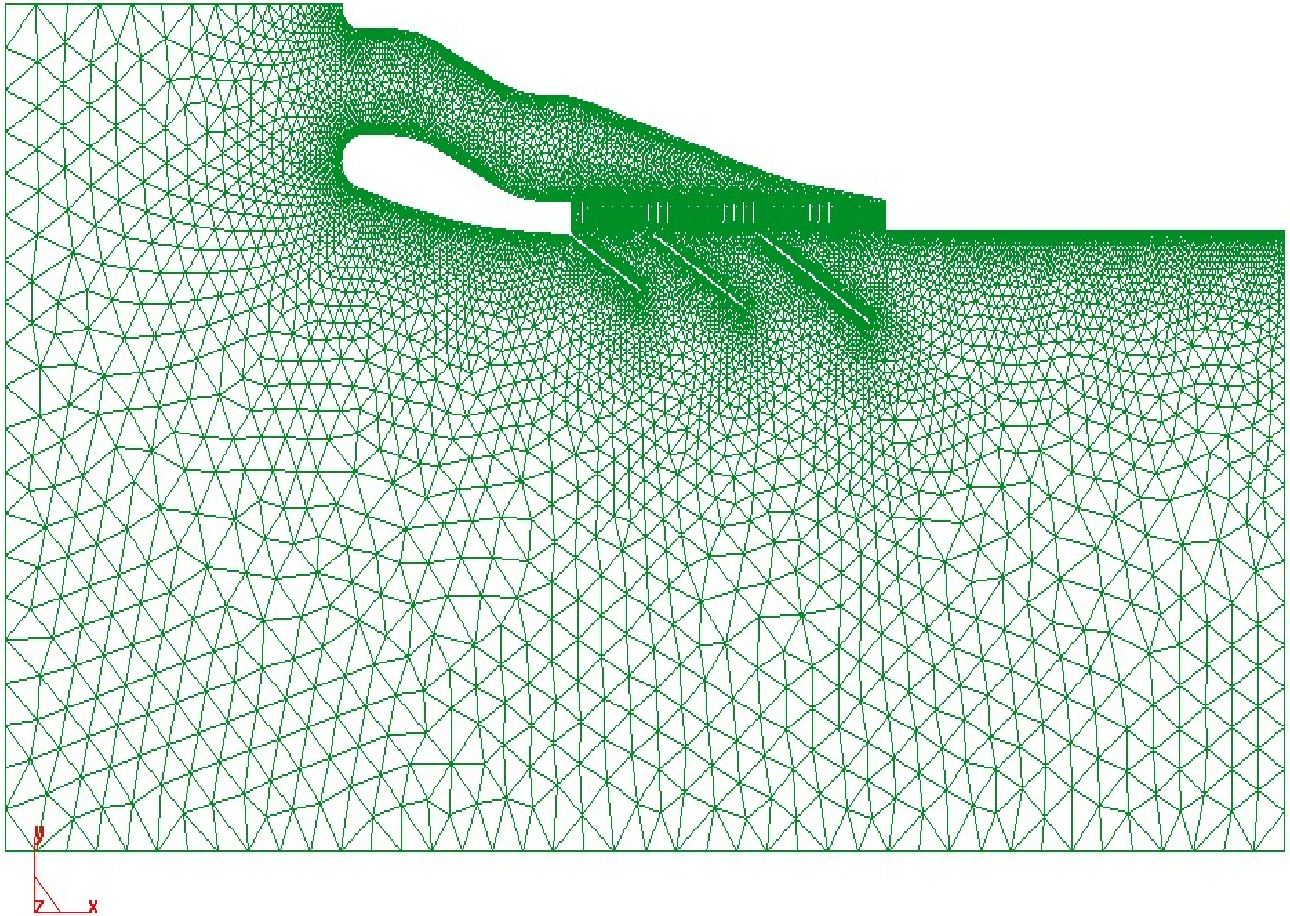
RETOS Y PERSPECTIVAS

- Geometry.

  We have seen that geometry enters through the notion of bicharacteristic rays for the control of continuous wave phenomena.

  But we have also seen that it enters in the Fourier space for filtering.

  The interaction of these two geometric aspects may be become rather complex, especially, for irtregular domains and meshes.



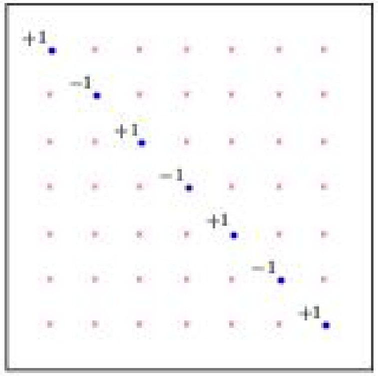
- More complex models

  We have analyzed purely conservative models like the wave equation. But nature is more complex: thermoelasticity, viscoelasticity,...

  The intrinsic dissipativity of these models has the tendency of damping out the spurious numerical high frequence components.

  But not enough to guarantee convergence at the control level!

- Unique continuation for the discrete Laplacian.



The eigenvector for the 5*−point finite-difference scheme for the Laplacian in the square, with eigenvalue* $\lambda = 4/h^2$*, taking values* ± 1 *along a diagonal, alternating sign and vanishing everywhere else in the domain.*

$$A_h \vec{\varphi} = \lambda \vec{\varphi}$$

$$\varphi_j = 0, \ \forall j \in \omega_h$$

$$\Rightarrow \varphi \equiv 0?$$

The problem arises in a much more general context: general geometries, finite elements, heat and wave equations,....

Generally speaking: What is the tool needed to analyze whether the fact that a solution of a discrete or semi-discrete system vanishes in a certain number of nodes, implies that the solution vanishes everywhere?

What is the discrete counterpart of Holmgren's Uniqueness Theorem or of Carleman's inequalities?

- Inverse Problems

  Inverse Problems are a mathematical topic related to environmental science, water pollution and they would also be of interest to any oil company. They can also arise when considering a medical question, or analysing medical equipment, for example, a medical specialist might want to know about tomography and scanning.

  `http://www.inverse-problems.com/`

- Shape design,...

  Recall that design optimization essentially combines mathematical optimization algorithms with engineering analysis models to generate designs with improved performance. In product development this approach is useful for products with a large number of interdependent design decisions or for new products where significant experience has not yet been accumulated. Current efforts are directed primarily towards complex and new technology products, and the augmentation of engineering analysis models with business or "enterprise" performance models, so that optimization results become more meaningful to management and the end users.

`http://ode.engin.umich.edu/research.html`

CONCLUSIONS:

- CONTROL AND NUMERICS DO NOT COMMUTE
- MUCH REMAINS TO BE DONE, BY COMBINING THE VARIOUS AREAS OF MATHEMATICS, TO PROVIDE A COMPLETE ANSWER TO THESE PROBLEMS AND A COMPLETE UNDERSTANDING OF ALL ITS CONSEQUENCES.