Chaotic synchronization on directed networks

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Abstract. The phenomenon of synchronization occurring in coupled chaotic maps on a directed random network is studied. The network is characterized by the average degree of its nodes and the fraction of directed links. It is found that the required coupling strength so that the chaotic synchronization emerges is smaller when the fraction of directed links is increased. In addition, the system undergoes a transition from an asynchronous phase to a synchronous one at some critical values of its parameters. The critical boundary separating the synchronous from the asynchronous regime is calculated on the parameter space of the system, given by the coupling strength and the fraction of directed links of the network. The phase transition between the two regimes is of second order for all values of the fraction of directed links, and the critical exponent depends of it.

The phenomenon of chaotic synchronization in dynamical networks is a well studied[1, 2, 3, 4]. There are many papers where dynamical units are defined on a undirected[5, 6], directed[7, 8] and complex[9, 10] networks. This phenomenon was also been studied by using continuous[11, 12] and discrete[13, 14] chaotic local dynamics. It is known that collective behavior of a system depends on properties of the substrate over which the dynamic takes place[15, 16]. In this paper, we explore how the directed links of the network affect the phenomenon of chaos synchronization. We consider a system of chaotic coupled maps defined on Erdös-Rényi random networks[17] of size N with a mean degree of links per node $\bar{k}$, and a fraction q of directed links. We show that there is a phase transition between chaotic spatiotemporal and synchronized phases, at a critical value of the parameter measuring the coupling strength between the units. This critical value of the coupling decreases when the fraction of directed links increases, and additionally the critical exponent of the transition also decreases.

Networks can be represented by their adjacency matrix $\mathbf{M}$, whose elements m_{ij} are given by

$$m_{ij} = \begin{cases} 1 & \text{if there is a link from node } i \text{ to node } j \\ 0 & \text{elsewhere} \end{cases} \quad , \qquad i, j = 1, 2, \ldots, N \ . \qquad (1)$$

The fraction of directed links q is defined as

$$q = 1 - \frac{\sum_{ij} m_{ij} m_{ji}}{\sum_{ij} m_{ij}} \quad , \quad i, j = 1, 2, \ldots, N \ . \qquad (2)$$

Let us consider a diffusively coupled map system on a connected random network with $N = 10^4$ nodes, described by

$$x_{t+1}^i = (1 - \epsilon) f(x_t^i) + \frac{\epsilon}{k_i} \sum_{j \in \nu_i} f(x_t^j) \quad , \quad i = 1, 2, \ldots, N \ ; \qquad (3)$$

where, x_t^i is the state of the ith element at discrete time t, ϵ is the coupling strength, ν_i is the set of neighbors of the ith element, k_i is the cardinality of ν_i, and $f(x_t)$ is a chaotic map that expresses the local dynamics. The mean degree of the network is given by

$$\bar{k} = \frac{1}{N} \sum_{i=1}^{N} k_i \, , \tag{4}$$

We set $\bar{k} = 9$. The local dynamic is given by the logarithmic map[18, 19],

$$f(x) = b + \ln(x) \, , \tag{5}$$

where b is the map parameter. We fix the parameter $b = -0.7$, that is, in the region where the map presents a robust chaotic regimen; The initial conditions are randomly chosen in the interval $[-10; 10]$.

The synchronized state is characterized by the asymptotic time-average $\langle \sigma \rangle$ of the instantaneous standard deviations σ_t of the distribution of map variables x_t^i, defined as

$$\sigma_t = \left[\frac{1}{N} \sum_{i=1}^{N} (x_t^i - \langle x_t \rangle)^2 \right]^{1/2} \, , \tag{6}$$

where $\langle x_t \rangle$ is the instantaneous mean of the values x_t^i. We discard the first $t = 10^4$ iterations and perform the time-average using the next 1000 iterates.

Figure 1 shows the asymptotic time-average of the instantaneous standard deviations $\langle \sigma \rangle$ as a function of the coupling strength ϵ and the fraction of directed links q.
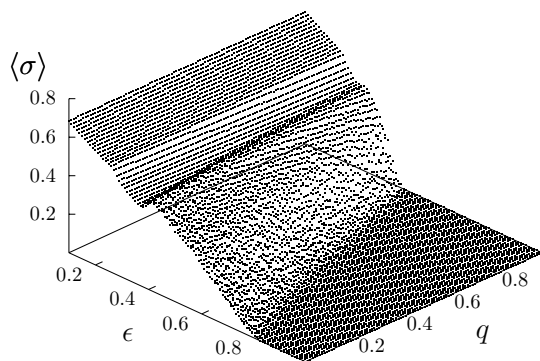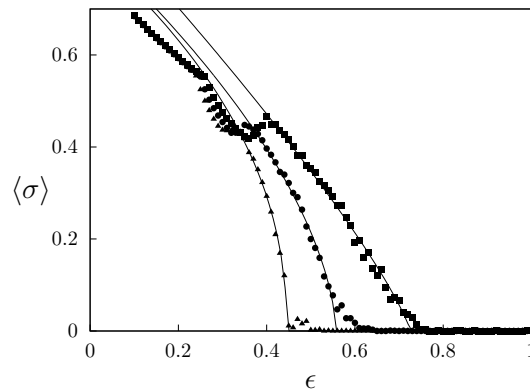

Figure 1. Asymptotic time-average of the instantaneous standard deviations $\langle \sigma \rangle$ as a function of the strength coupling ϵ and the fraction of directed links q; with $N = 10^4$ and $\bar{k} = 9$. All points are averaged over 6 realizations.

Figure 2. Asymptotic time-average of the instantaneous standard deviations $\langle \sigma \rangle$ as a function of the strength coupling ϵ. Squares $q = 0.1$, circles $q = 0.5$ and triangles $q = 0.9$. The continuous lines represent the fit of the equation (7) close to ϵ_c.

Note that, when the coupling strength ϵ increases, there is a transition in the system from a chaotic spatiotemporal ($\langle\sigma\rangle > 0$) to a synchronized ($\langle\sigma\rangle = 0$) phase. In figure 2 it is shown the asymptotic time-average of the instantaneous standard deviations $\langle\sigma\rangle$ as a function of the coupling strength ϵ with $q = 0.1$, $q = 0.5$ and $q = 0.9$.

The variation of the order parameter $\langle\sigma\rangle$ near the critical value of coupling strength ϵ_c can be characterized by a critical exponent β as

$$\langle\sigma\rangle = (\epsilon_c - \epsilon)^\beta . \tag{7}$$

We have calculated numerically the critical values of the coupling strength ϵ_c for the onset of synchronization in directed random network. The continuous lines in Figure 2 are the fitting of equation (7) for the given values of q.

Figure 3 shows the resulting critical boundary ϵ_c for the transition from chaotic spatiotemporal to synchronized regime, as well as the phase diagram of the system in the parameter space (ϵ, q).
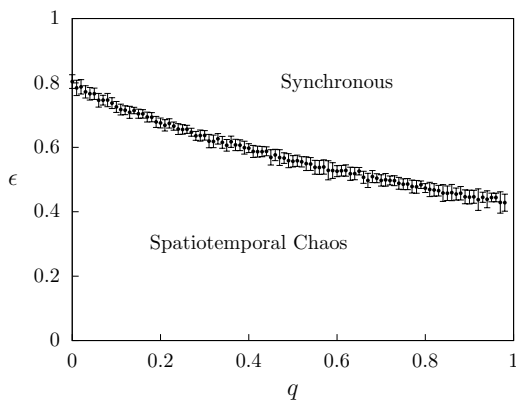

Figure 3. Phase diagram in the parameter space (ϵ, q). The numerically calculated critical boundary ϵ_c is shown with its error.
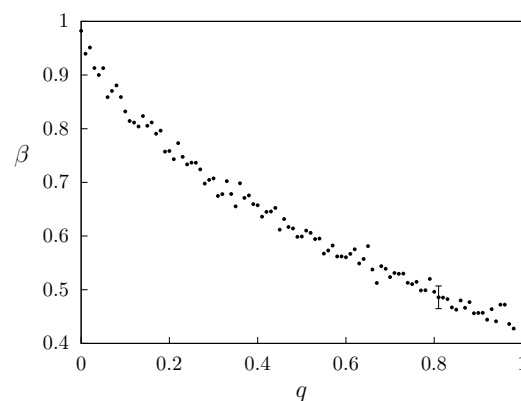
Figure 4. Numerically calculated critical exponent β as a function of q with a typical error bar.

Note that the critical value of the coupling strength ϵ_c defines a boundary in the phase diagram (ϵ, q) between the two regimes. Also, it can appreciate that the strength coupling necessary to synchronize the system is smaller when the fraction of directed links is increased.

Finally, Figure 4 shows the relation between the critical exponent β and the fraction of directed links q. It can see that, as q increases, the exponent β becomes smaller and the corresponding phase transition from spatiotemporal chaos to synchronized behavior gets more abrupt.

In summary, we have studied how the directed links affect the synchronization phenomenon on a Erdös-Rényi random networks. Although we use one class of networks and a simple dynamic for the nodes, we expect that the essential properties of the transition between synchronized-chaotic spatiotemporal phases is captured by this model.

By varying the fraction of directed links in the random networks, the behavior of the transition to synchronization can be studied in the regime between undirected lattices and completely directed random networks. The critical boundary separating synchronous and spatiotemporal chaos regimes was calculated on the parameter space (ϵ, q) of the system. We have found that

the character of this transition is always of second order, that is, the critical exponent that characterize the transition is in the interval $\beta \in (0, 1)$ for all values of fraction of directed links q. The form how a phenomena appear depends on properties of the substrate over which the dynamic is carried out.

In this case, the spatiotemporal chaos-synchronous phase transition on a Coupled Chaotic Maps Lattice depends of the fraction of directed links of the network. In order to the synchronization phenomenon arises in the system, a smaller coupling strength, ϵ, is required when the fraction of directed links of the network, q, increases; and the phase transition becomes sharper.

Acknowledgments

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