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Collective behavior of coupled chaotic maps

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Abstract

The collective behavior of a coupled map lattice having *unbounded* chaotic local dynamics is investigated through the properties of its mean field. The presence of unstable periodic orbits in the local maps determines the emergence of nontrivial collective behavior. Windows of collective period-two states are found in parameter space.

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Coupled map lattices (CMLs) have provided fruitful models for the study of many spatiotemporal processes in a variety of contexts (for recent surveys, see Ref. [1]). Recently, there has been great interest in the use of CML models in the investigation of global behaviors of networks of chaotic elements [2–4]. Such collective phenomena have many implications ranging from the fundamentals of statistical mechanics [5] to biological information processing, and even possible practical applications [6].

In this Letter, we study the collective behavior of the one-dimensional CML system

$$x_{t+1}(i) = f(x_t(i)) + \gamma\left[f(x_t(i-1)) + f(x_t(i+1)) - 2f(x_t(i))\right], \quad (1)$$

where t is a discrete time, i labels the lattice sites ($i = 1, \ldots, N$), γ is a parameter measuring the diffusive coupling strength between neighboring sites, and f is a map describing the local dynamics. Periodic boundary conditions are assumed.

The collective behavior of the lattice can be monitored through the instantaneous mean field or *activity* of the network, defined as the space average of the local variables $x(i)$ at time t [2],

$$S_t = \frac{1}{N} \sum_{i=1}^{N} x_t(i). \quad (2)$$

A wide variety of local map functions f have been employed in CML models depending on the particular application. Usually, bounded maps belonging to some universality class are considered to be the source of local chaos in the study of collective spatiotemporal dynamics in CMLs with local [2,3] or global couplings [4]. Here we investigate the collective behavior of the lattice described by (1) having unbounded local dynamics such as the logarithmic map [7]

$$f(x) = b + \ln|x|. \quad (3)$$

This map posseses no maximum or minimum and its Schwarzian derivative is always positive. Two stable fixed points satisfying $f(x^*) = x^*$ exist for this map: $x_1^* < -1$, for $b < -1$, which becomes unstable at $b = -1$; and $x_2^* > 1$, for $b > 1$, which arises

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from a tangent bifurcation at $b = 1$. Chaos occurs in the parameter region $b \in [-1, 1]$. There are no gaps separating chaotic bands at any given value of $b \in [-1, 1]$ and no periodic windows appear in any subinterval of b in this range [7]. These special features, contrasting with the universal properties of the commonly used local maps, make maps of the type (3) an alternative tool for exploring the essential requirements for the emergence of collective behavior in CMLs and the generality of this phenomenon.

The dynamics of the CML (1) can be written in vector form as

$$\boldsymbol{x}_{t+1} = \boldsymbol{f}(\boldsymbol{x}_t) + \gamma \mathbf{M} \boldsymbol{f}(\boldsymbol{x}_t), \tag{4}$$

where the N-dimensional vectors $\boldsymbol{x}_t$ and $\boldsymbol{f}(\boldsymbol{x}_t)$ have components $[\boldsymbol{x}_t]_i = x_t(i)$ and $[\boldsymbol{f}(\boldsymbol{x}_t)]_i = f(x_t(i))$, respectively; and $\mathbf{M}$ is an $N \times N$ tridiagonal, symmetric matrix expressing diffusive coupling among the components $[\boldsymbol{x}_t]_i$. The non-zero components of $\mathbf{M}$ are $\mathbf{M}_{ii} = -2$; $M_{ij} = 1$ $(i = j \pm 1)$.

With the local map (3), the stable, spatially homogeneous, stationary states $x_t(i) = x_1^*$, for $b < -1$; or $x_t(i) = x_2^*$, for $b > 1$, $\forall i$, are possible for the system (1). The linear stability analysis of these states leads to the bifurcation conditions

$$(1 + \gamma \mu_k) f'(x_{1,2}^*) = \pm 1, \tag{5}$$

where $\{\mu_k : k = 1, \ldots, N\}$ is the set of eigenvalues of the coupling matrix $\mathbf{M}$ which satisfy $\mu_k \in [-4, 0]$ [8]. Eqs. (5) yield boundary curves in the parameter plane (γ, b) which determine where each homogeneous, stationary state can be observed. Fig. 1 shows the first stability boundaries for each state, corresponding to $\mu_k = -4$. The state $x_t(i) = x_1^*$, $\forall i$, is stable for parameter values in the region enclosed by the r.h.s. ± 1 boundaries corresponding to x_1^* in Eq. (5), with $b < -1$. Similarly, the state $x_t(i) = x_2^*$, $\forall i$, is stable inside the parameter region bounded by the r.h.s. ± 1 curves associated to x_2^*, with $b > 1$. Within these regions of stable homogeneous stationary states, the asymptotic values of the mean field are $S_t = x_1^*$ and $S_t = x_2^*$, respectively. The crossing of either boundary signals the appearance of a spatially inhomogeneous state which should be reflected in a dispersion of S_t.

When the value of the parameter b is in the range $[-1, 1]$, corresponding to local chaotic dynamics, the asymptotic collective behavior of the lattice, as given by $S_{t \to \infty}$, reveals the existence of global low-dimensional periodic attractors, subjected to fluctuations of intrinsic statistical origin. Fig. 2 shows the

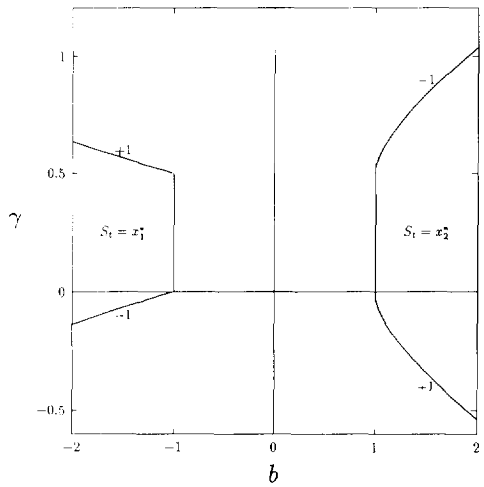

Fig. 1. Stability boundaries for the homogeneous stationary states of the CML (1). The labels on each curve identify the first stability boundaries with ± 1 in Eq. (5) for each state, corresponding to $\mu_k = -4$. The vertical lines at $b = -1$ and $b = 1$ correspond to $\mu_k = 0$. Curves associated with other eigenvalues lie outside the regions enclosed by the boundaries shown. The states $S_t = x_1^*$ and $S_t = x_2^*$ are stable inside the indicated regions.

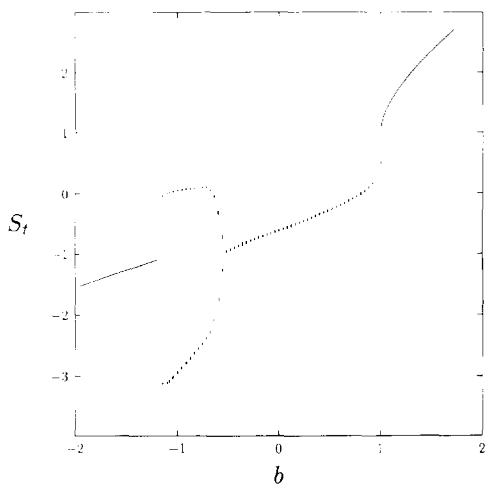

Fig. 2. Bifurcation diagram of the asymptotic S_t, as a function of the local parameter b. Coupling is fixed at $\gamma = 0.5$. For each value of b, 100 iterates are shown, after discarding 5000 transients. Lattice size is $N = 10^5$. For $b < -1$ and $b > 1$, S_t corresponds to the values of the homogeneous stationary states x_1^* and x_2^*, respectively.

bifurcation diagram of the asymptotic mean field S_t as a function of the local parameter b. The quantity S_t was calculated at each time step during a run starting from random initial conditions uniformly distributed on the interval $[-8, 4]$ at each site and for each value of b, after discarding the transients. The coupling strength was fixed at $\gamma = 0.5$, corresponding to a "totalistic" coupling of the "game of life" type [2]. Similar bifurcation diagrams can be obtained for other fixed values of the coupling parameter.

For $b < -1$ or $b > 1$, the asymptotic mean field is identical to the values of the corresponding fixed points of the single map (3) in these ranges of b, as expected from the stability diagram of the homogeneous stationary states in Fig. 1. In the region $b \in [-1, 1]$, Fig. 2 shows a pitchfork bifurcation at a value $b_c \simeq -0.52$ from a collective fixed point state (a state for which the time series of S_t manifestly displays statistical fluctuations around a single value) to a collective period-two state (a state for which the time series of S_t alternatively varies between the corresponding neighborhoods of two separated values). The small vertical segments seen in Fig. 2 can be interpreted as the amplitude of the fluctuations of the mean field about the corresponding global attractor at given parameter values. The fluctuations around the global stationary or the period-two attractors are due to the fact that the local variables behave chaotically, as can be attested by the time series of any site and by the existence of a positive largest Lyapunov exponent.

The appearance of a period-two collective state in the one-dimensional lattice (1) is related to the fact that the iterates of the local map (3) move alternatively from values above the unstable fixed point x_1^* to values below this point in the interval $b \in [-1, b_c]$, even though there are no separated chaotic bands [9]. The iterates behave chaotically within each side of the unstable fixed point x_1^*, but *periodically* about it. The unstable fixed point x_1^* establishes a "symmetry" line around which the iterates oscillate with period two. For comparison, such an effect does not occur in the unbounded map $g(x) = a - 1/x$, which lacks unstable periodic orbits for $a \in [-2, 2]$; consequently, no collective periodic behavior emerges in a coupled lattice of these maps at that range of the parameter a [9].

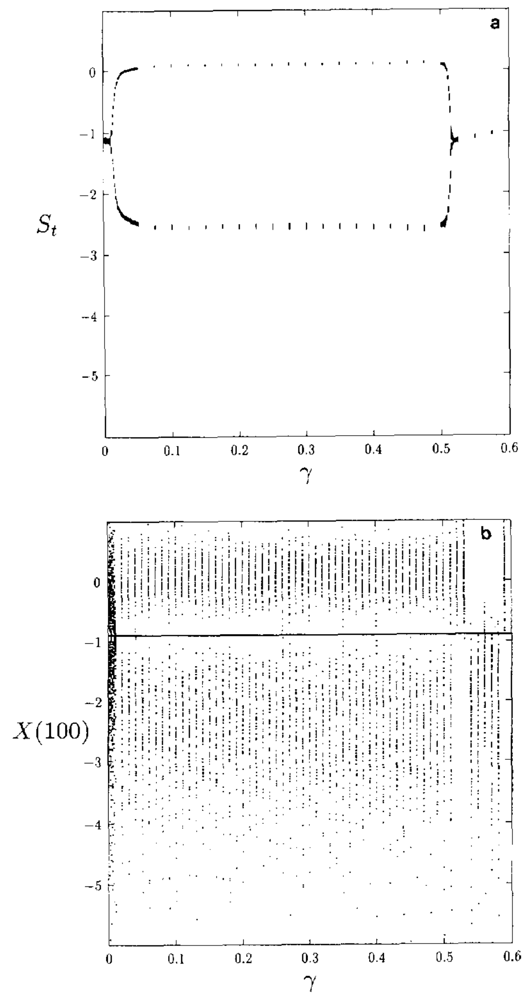

Fig. 3. (a) S_t as a function of the coupling parameter γ. Local parameter is fixed at $b = -0.8$. For each value of γ, 100 iterates are shown, after discarding 5000 transients. Lattice size is $N = 10^5$. (b) Asymptotic iterates of site $i = 100$, for the same parameters as in (a). The horizontal line corresponds to the value of the unstable x_1^* for the given b.

Coupling induces synchronization of the array in the range $b \in [-1, b_c]$, in the sense that iterates of the local chaotic sites tend to move together above and below the unstable fixed point x_1^* at alternate time steps. The individual local values may be different within each of these two regions at a given instant. Fig. 3a presents the asymptotic mean field S_t as a function of the coupling strength γ, keeping constant $b = -0.8$. When γ is close to zero, and starting each time from random initial conditions, S_t

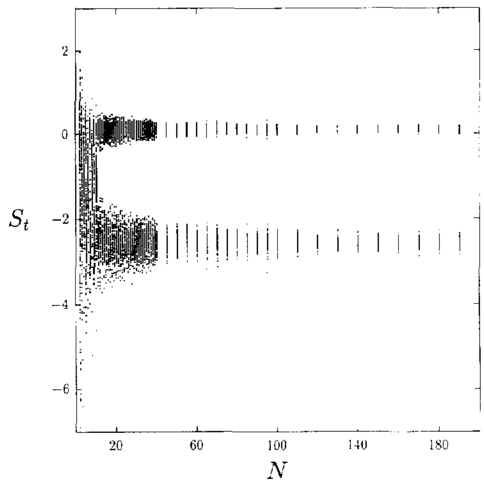

Fig. 4. Asymptotic mean field S_t as a function of lattice size N. Fixed parameters are $\gamma = 0.5; b = -0.8$.

fluctuates about an average value of the desynchronized local maps. There exists a critical value of the coupling at $\gamma_c \simeq 0.01$ where the transition from a statistical average to a period-two collective state takes place. The periodic behavior of S_t remains practically unchanged until the coupling reaches a second critical value $\gamma_c' \simeq 0.53$ at which synchronization is again lost. Fig. 3b shows the asymptotic behavior of an arbitrary individual site simultaneously monitored as a function of the coupling. The value of the unstable fixed point x_1^* corresponding to $b = -0.8$ is indicated by a horizontal line. Coupling enhances the separation of alternate iterates about x_1^*, creating a gap between them for $\gamma \in [\gamma_c, \gamma_c']$. The mean field reflects this separation as well as the synchronization of the lattice produced by the coupling.

Fig. 4 shows the dependence of the asymptotic S_t on the size of the lattice N, with the other parameters held fixed ($\gamma = 0.5$, $b = -0.8$). There is a rather small critical size $N_c \simeq 10$ at which the global period-two attractor distinctively emerges. As in Fig 2, the vertical segments represent fluctuations around the global period-two attractor and they correspond to even or odd steps of the asymptotic time series of S_t, respectively. The variance of either subset of steps of the time series of S_t decreases as N^{-1}, as expected. However, the variance of the mean field itself, for the chosen parameters, tends to a constant value for large enough N. In the limit $N \to \infty$, the global period-two orbit will have the form S_t: $\ldots s_1, s_2, s_1, s_2 \ldots$. The time average of the mean field will be $\bar{S} = (s_1 + s_2)/2$, while the variance σ will yield

$$\sigma = \frac{1}{T} \sum_{t=1}^{T} \left(S_t - \bar{S}\right)^2 = \frac{1}{T} \sum_{t=1}^{T} \frac{(s_1 - s_2)^2}{4}. \qquad (6)$$

For the parameters of Fig. 4, $s_1 \simeq -2.55$, $s_2 \simeq 0.08$. The limiting value $\sigma \simeq 1.72$ is approached for $N \simeq 10^3$. The existence of a saturation value for the variance of the mean field characterizes the emergence of nontrivial (i.e. periodic, quasiperiodic, or chaotic) collective behavior. This phenomenon has been called "violation of the law of large numbers" in the context of globally coupled maps belonging to some universality class (tent, quadratic, or circle maps) [4,10].

The onset of a periodic collective state at some values of the parameters of the system is reminiscent of a phase transition, because it corresponds to abrupt changes in the statistical properties of the lattice, as described by $S_{t \to \infty}$, a quantity which acts as an order parameter. We have found windows of global periodic behavior in the space of parameters. Periodic collective states have been observed in higher dimensional lattices of locally chaotic coupled maps. In those cases, the collective behavior consists in statistical cycling of the mean field (or the probability density) among chaotic bands of the local one-hump maps [2,3,11]. This paper shows that high space dimension, large system size, strong coupling, bounded iterates, gap-separated chaotic bands, or the existence of periodic windows in the local dynamics, are not essential requirements for the emergence of nontrivial collective behavior. We have found that coupling can synchronize the cycling of the chaotic iterates and enhances their separation around unstable periodic orbits of the local maps in some parameter ranges giving rise to periodic global behavior in a lattice. Our results suggest that the emergence of nontrivial collective behavior should be a rather generic phenomenon in deterministic systems of coupled chaotic elements, where unstable periodic orbits are always present.

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